Negatons, positons, rational-like solutions and conservation laws of the Korteweg–de Vries equation with loss and non-uniformity terms

2004 J. Phys. A: Math. Gen. 37 851

(http://iopscience.iop.org/0305-4470/37/3/021)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **37** (2004) 851–865 PII: S0305-4470(04)67327-6

Negatons, positons, rational-like solutions and conservation laws of the Korteweg–de Vries equation with loss and non-uniformity terms

Da-jun Zhang and Deng-yuan Chen

Department of Mathematics, Shanghai University, Shanghai 200436, People's Republic of China

E-mail: djzhang@mail.shu.edu.cn

Received 7 August 2003, in final form 3 October 2003
Published 7 January 2004
Online at stacks.iop.org/JPhysA/37/851 (DOI: 10.1088/0305-4470/37/3/021)

Abstract
Solitons, negatons, positons, rational-like solutions and mixed solutions of a non-isospectral equation, the Korteweg–de Vries equation with loss and non-uniformity terms, are obtained through the Wronskian technique. The non-isospectral characteristics of the motion behaviours of some solutions are described with some figures made by using Mathematica. We also derive an infinite number of conservation laws of the equation.

PACS number: 05.45.Yv

1. Introduction

As a typical type of singular solution, the positons of the Korteweg–de Vries (KdV) equation were first considered by Matveev [1]. Positons correspond to positive eigenvalues of the Schrödinger spectral problem. Matveev obtained this type of solution from the generalized Wronskian solutions derived from the Darboux transformation. On the basis of Matveev's idea, positons of some other equations have been discussed successively [2–4]. Another type of singular solution of the KdV equation, called negatons (corresponding to negative eigenvalues of the Schrödinger spectral problem), can also be derived from Matveev's generalized Wronskian solutions. In 1996, Rasinariu *et al* [5] gave a systematic classification and a detailed discussion of the structure and motion of negatons and positons of the KdV equation. Recently, Zeng *et al* [6] obtained negatons and positons of some soliton equations with self-consistent sources.

The Hirota method [7] and Wronskian technique [8] are two efficient direct ways to find soliton solutions for nonlinear evolution equations. The advantage of the Wronskian technique is that this approach admits direct verifications of solutions, and solutions in such a determinantal form are easy for the discussion of some related properties. There are some generalizations about this technique. For example, it can be generalized to find rational solutions and mixed soliton–rational solutions in the Wronskian form for few soliton equations [9, 10] by following the idea of long wave limitation [11]. Recently, we also developed some new determinantal identities to obtain exact solutions for some soliton equations with self-consistent sources [12–14]. Another generalization is to alter the conditions which the entries of a Wronskian satisfy. In this way, in 1988 Sirianunpiboon *et al* [15] derived a more general Wronskian solution for the KdV equation. Their result provides us more choices of entries of a Wronskian. They could not only give the soliton solution [8] and rational solution [9] in the Wronskian form, but also obtained a generalized Wronskian form which later Matveev derived from the Darboux transformation to generate positons [1]. In a very recent paper, Ma discussed in detail a bridge going from Wronskian solutions to generalized Wronskian solutions of the KdV equation [16]. He also considered the so-called generalized positons and negatons.

In this paper, we generalize the Wronskian technique to the following KdV equation with loss and non-uniformity terms [17, 18]

$$u_t + 2\alpha u + (c_0 + 6\beta\,\mathrm{e}^{-2\alpha t} + \alpha x)u_x + 6uu_x + u_{xxx} = 0 \tag{1.1}$$

where α, β and c_0 are real constants and we assume that u tends to 0 as $x \to \pm\infty$. This equation can describe soliton waves in a certain type of non-uniform media with a relaxation effect [17]; α describes the relaxation and non-uniformity of the media and β and c_0 are related to wave velocity. Taking $\beta = 0$, equation (1.1) is just the equation considered by Hirota and Satsuma in 1976 [17]. If we employ a new potential

$$v = u + \beta\,\mathrm{e}^{-2\alpha t} \tag{1.2}$$

equation (1.1) turns out to be

$$v_t + 2\alpha v + (c_0 + \alpha x)v_x + 6vv_x + v_{xxx} = 0 \tag{1.3a}$$

with time-varying non-vanishing boundary condition

$$v|_{x\to\pm\infty} = \beta\,\mathrm{e}^{-2\alpha t}. \tag{1.3b}$$

Chan and Li [18] have investigated equation (1.3) with variable coefficients through the inverse scattering transform and have shown soliton characteristics of the related solutions.

Similar to [17] and [18], we can give the Lax pair of equation (1.1), i.e.,

$$\phi_{xx} = (\lambda - u)\phi \tag{1.4a}$$

$$\phi_t = -u_x\phi - (4\lambda + 2u + c_0 + 6\beta\,\mathrm{e}^{-2\alpha t} + \alpha x)\phi_x \tag{1.4b}$$

where the spectral parameter λ follows

$$\lambda_t = -2\alpha\lambda. \tag{1.5}$$

We can see that equation (1.1) is a non-isospectral evolution equation in terms of λ.

Equation (1.1) is one of the so-called x-coefficient equations which can describe nonlinear waves in non-uniformity media [19]. Such types of nonlinear evolution equations with non-isospectral auxiliary linear problems have been widely considered, for example, in [20]. In general, these equations can be exactly integrable [21]. However, whether they have any singular solutions, such as negatons, positons and rational solutions, and if they have, what the non-isospectral characteristics of these solutions are, have not been investigated so far.

One of the purposes of this paper is to show that equation (1.1) admits solitons, negatons, positons, rational-like solutions and mixed solutions in the Wronskian (or generalized Wronskian) form, although it is an equation with non-isospectral properties. We obtain the solutions by generalizing the procedure proposed by Sirianunpiboon *et al* [15] to this non-isospectral equation. Some of the obtained solutions are novel.

Then we hope to describe some non-isospectral characteristics of these special solutions with non-isospectral properties and make comparisons between non-isospectral and isospectral characteristics. The behaviours of some solutions are shown with some figures made by using Mathematica. An obvious and common characteristic of these solutions is that the amplitude and velocity vary with time t. This results from the fact that the spectral parameter λ varies with time. We hope to provide more information about those nonlinear waves with non-isospectral properties.

We also derive an infinite number of conservation laws of the KdV equation with loss and non-uniformity terms by regarding it as an isospectral equation.

This paper is organized as follows. In section 2, we solve equation (1.1) by means of the Hirota method and N-soliton solutions are obtained. In section 3, solutions in the Wronskian form are verified. In section 4, solitons, positons, negatons, rational-like solutions and mixed solutions are discussed and some figures are given. Finally, in section 5, the conservation laws are derived.

2. N-soliton solution in terms of the Hirota method

The bilinear form of equation (1.1) is given by

$$\left[D_tD_x + D_x^4 + (c_0 + 6\beta\,\mathrm{e}^{-2\alpha t} + \alpha x)D_x^2\right]f\cdot f + 2\alpha f_x f = 0 \tag{2.1}$$

through the following transform

$$u = 2(\ln f)_{xx} \tag{2.2}$$

where D is the well-known Hirota bilinear operator defined by

$$D_x^m D_t^n a\cdot b = (\partial_x - \partial_{x'})^m(\partial_t - \partial_{t'})^n a(x,t)b(x',t')|_{x'=x,t'=t}.$$

Similar to [17], we can derive one- and two-soliton solutions of equation (1.1) and further conjecture that N-soliton solutions follow the form

$$f = \sum_{\mu=0,1}\exp\left\{\sum_{j=1}^{N}\mu_j\eta_j + \sum_{1\leqslant j<l}^{N}\mu_j\mu_l A_{jl}\right\} \tag{2.3a}$$

$$\eta_j = xp_j(t) + \frac{c_0}{\alpha}[p_j(t) - q_j] + \frac{1}{3\alpha}\left[p_j^3(t) - q_j^3\right] + 2\frac{\beta}{\alpha}[\mathrm{e}^{-2\alpha t}p_j(t) - q_j] + \eta_j^{(0)} \qquad p_j(t) = q_j\mathrm{e}^{-\alpha t}, q_j = p_j(0) \tag{2.3b}$$

$$e^{A_{jl}} = \left(\frac{q_j - q_l}{q_j + q_l}\right)^2 \tag{2.3c}$$

where $\eta^{(0)}$ is a real constant, and the sum over $\mu = 0, 1$ refers to each μ_j equals 0 or 1, $j = 1, 2, \ldots, N$. We will show this conjecture to be valid in section 4.

3. Solutions in the Wronskian form

In this section, we derive solutions in the Wronskian form for equation (1.1). We show that the process proposed by Sirianunpiboon *et al* [15] can also apply to this non-isospectral equation.

Let us first specify the results of Sirianunpiboon *et al* [15]. The KdV equation is

$$u_t + 6uu_x + u_{xxx} = 0. \tag{3.1a}$$

Its bilinear form

$$\left(D_t D_x + D_x^4\right) f \cdot f = 0 \qquad (u = 2(\ln f)_{xx}) \tag{3.1b}$$

admits the $N \times N$ Wronskian solution [8]

$$f = W(\psi_1, \psi_2, \ldots, \psi_N) = \begin{vmatrix} \psi_1 & \psi_1^{(1)} & \cdots & \psi_1^{(N-1)} \\ \psi_2 & \psi_2^{(1)} & \cdots & \psi_2^{(N-1)} \\ \vdots & \vdots & \cdots & \vdots \\ \psi_N & \psi_N^{(1)} & \cdots & \psi_N^{(N-1)} \end{vmatrix} = |0, 1, \ldots, N-1| = |\widehat{N-1}| \tag{3.2}$$

in which $\psi_j^{(l)}(x,t) = \partial^l \psi_j(x,t)/\partial x^l$ and each ψ_j satisfies

$$\psi_{j,xx} = k_j^2 \psi_j \tag{3.3}$$

$$\psi_{j,t} = -4\psi_{j,xxx} \tag{3.4}$$

where each k_j is a real constant. Here, following the notation of Freeman and Nimmo [8], let $\widehat{N-j}$ indicate the set of consecutive columns $0, 1, 2, \ldots, N-j$.

In 1988, Sirianunpiboon *et al* [15] proved that the above Wronskian still satisfied the KdV bilinear equation (3.1b) if the condition (3.3) was replaced by

$$\psi_{j,xx} = \sum_{i=1}^{j} b_{ji}(t)\psi_j \tag{3.5}$$

where each $b_{ji}(t)$ is an arbitrary function of t. Such a generalization can provide more exact solutions for the KdV equation (3.1a).

In this paper, for equation (1.1), we have the following result.

Theorem 1. *The bilinear equation (2.1) admits Wronskian solution (3.2) where* $\{\psi_j\}$ *enjoy the conditions of (3.5) and*

$$\psi_{j,t} = -4\psi_{j,xxx} - (c_0 + 6\beta\,\mathrm{e}^{-2\alpha t} + \alpha x)\psi_{j,x}. \tag{3.6}$$

It is not difficult to verify this result. The verification is similar to the procedure in [8] and [15] except the expressions for the derivatives of f with respect to t. For example, following the condition (3.6), we have

$$\begin{aligned} f_t = {} & -4(|\widehat{N-4}, N-2, N-1, N| - |\widehat{N-3}, N-1, N+1| + |\widehat{N-2}, N+2|) \\ & - (c_0 + 6\beta\,\mathrm{e}^{-2\alpha t} + \alpha x) f_x - \frac{\alpha N(N-1)}{2} f \end{aligned}$$

and

$$\begin{aligned} f_{xt} = {} & -4(|\widehat{N-5}, N-3, N-2, N-1, N| - |\widehat{N-3}, N, N+1| + |\widehat{N-2}, N+3|) \\ & - (c_0 + 6\beta\,\mathrm{e}^{-2\alpha t} + \alpha x) f_{xx} - \alpha f_x - \frac{\alpha N(N-1)}{2} f_x. \end{aligned}$$

We will show in the following section that the Wronskian (3.2) with conditions (3.5) and (3.6) can present us more exact solutions for equation (1.1).

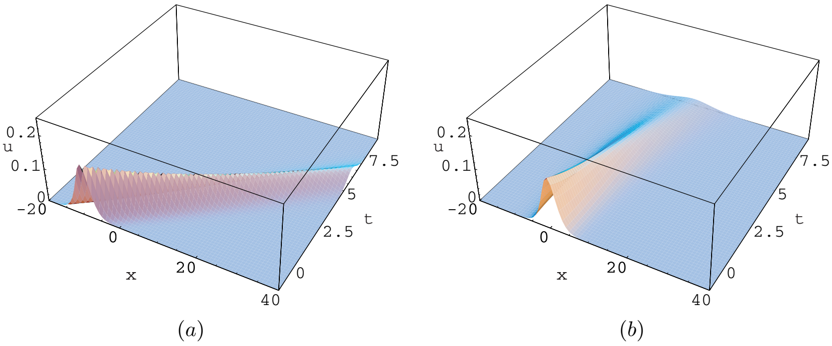


Figure 1. The shape and motion of the one-soliton solution as given by equation (4.3) for $\eta_1^{(0)} = 0, q_1 = 0.7, \alpha = 0.1, c_0 = 1, x \in [-25, 40]$ and $t \in [-1.2, 9]$: (*a*) $\beta = 1$; (*b*) $\beta = -0.25$.

4. Applications

4.1. Solitons

We have shown in the preceding section that the bilinear equation (2.1) has a Wronskian solution (3.2) where the entries enjoy more general conditions (3.5) and (3.6). Solitons can be obtained by directly setting

$$\psi_j = \mathrm{e}^{\frac{\eta_j}{2}} + (-1)^{j-1}\,\mathrm{e}^{-\frac{\eta_j}{2}} \qquad j = 1, 2, \ldots, N \tag{4.1}$$

where η_j is defined by equation (2.3*b*). In this case, ψ_j satisfies equation (3.6) and

$$\psi_{j,xx} = \tfrac{1}{4} p_j^2(t)\psi_j. \tag{4.2}$$

Here we note that, assuming that $0 < q_1 < q_2 < \cdots < q_N$ and employing the similar procedure given in [12, 13], we can show that the above soliton solution in the Wronskian form can be written as

$$f = \left(\frac{1}{2}\right)^{\frac{N(N-1)}{2}} \left(\prod_{j=1}^{N} \mathrm{e}^{-\frac{\eta_j}{2}}\right)\left(\prod_{1\leqslant j<l\leqslant N}(p_l(t) - p_j(t))\right)\sum_{\mu=0,1}\exp\left\{\sum_{j=1}^{N}\mu_j\bar{\eta}_j + \sum_{1\leqslant j<l\leqslant N}\mu_j\mu_l A_{jl}\right\}$$

where $\bar{\eta}_j = \eta_j - \frac{1}{2}\sum_{l=1,l\neq j}^{N} A_{jl}$, which are the same as the solution (2.3) in Hirota form in terms of recovering a solution of equation (1.1) from the transformation (2.2). Such uniformity is a kind of verification of the N-soliton solution given by equation (2.3).

The simplest case, i.e., one-soliton solution, is

$$u = \frac{1}{4} q_1^2\,\mathrm{e}^{-2\alpha t}\,\mathrm{sech}^2\frac{\eta_1}{2}. \tag{4.3}$$

The shape and motion of the solution is shown in figure 1. For any fixed time t, the wave shows the characteristic of $u \to 0$ as $x \to \pm\infty$. However, the amplitude of the wave, i.e. $\frac{1}{4}q_1^2\,\mathrm{e}^{-2\alpha t}$, tends to zero as $t \to +\infty$. This non-isospectral characteristic is caused directly by the dominant term $\mathrm{e}^{-\alpha t}$ in equation (4.3) as $t \to +\infty$ where we take $\alpha > 0$. So, the so-called soliton solutions named by Hirota and Satsuma [17] are essentially amplitude-decay solitary waves. Apart from the time-dependent wave amplitude, the wave velocity is not constant. For solution (4.3), the velocity of the vertex of the wave (which corresponds to $\eta_1 = 0$) is

$$\frac{\mathrm{d}}{\mathrm{d}t}x(t) = \alpha x + c_0 + \left(6\beta + q_1^2\right)\mathrm{e}^{-2\alpha t}.$$

We can find that velocity depends on space, time and the parameters α, β, c_0 and q_1. In some case, the absolute value of the velocity can be very small, which suggests the wave runs very slowly as shown in figure 1(*b*) where the wave seems to be actionless except that its amplitude decays. The velocity can also be negative; at that time the wave goes in the negative x-direction.

4.2. Negatons and positons

We first derive a generalized Wronskian form which still satisfies the bilinear equation (2.1) as well. Consider the Wronskian (3.2) where each ψ_j enjoys (3.6) and

$$\psi_{j,xx}(q_j) = \pm\tfrac{1}{4}q_j^2\,\mathrm{e}^{-2\alpha t}\psi_j(q_j) \tag{4.4}$$

and we suppose that each ψ_j is differentiable with respect to q_j for any order. Then we have the following Taylor expansion

$$\psi_j(q_j+\delta) = \sum_{m=1}^{+\infty}\frac{1}{m!}G_m(x,t,q_j)\delta^m \tag{4.5}$$

where

$$G_m = \frac{\partial^m}{\partial^m_{q_j}}\psi_j(q_j) = \partial^m_{q_j}\psi_j(q_j). \tag{4.6}$$

It is easy to find that $\{G_m\}$ meets

$$G_{m,t} = -4G_{m,xxx} - (c_0 + 6\beta\,\mathrm{e}^{-2\alpha t} + \alpha x)G_{m,x} \tag{4.7a}$$

and

$$\begin{aligned}
G_{0,xx} &= \pm\frac{1}{4}\,\mathrm{e}^{-2\alpha t}q_j^2 G_0\\
G_{1,xx} &= \pm\frac{1}{4}\,\mathrm{e}^{-2\alpha t}\big[q_j^2G_1 + 2q_jG_0\big]\\
G_{m,xx} &= \pm\frac{m!}{4}\,\mathrm{e}^{-2\alpha t}\left[\frac{1}{m!}q_j^2G_m + \frac{2}{(m-1)!}q_jG_{m-1} + \frac{1}{(m-2)!}G_{m-2}\right] \qquad (m\geqslant 2).
\end{aligned} \tag{4.7b}$$

So, according to theorem 1,

$$f = W(G_0, G_1, \ldots, G_s) = W\big(\psi_j, \partial_{q_j}\psi_j, \ldots, \partial^s_{q_j}\psi_j\big) \tag{4.8}$$

is a solution of equation (2.1). A more general solution can be denoted by

$$f = W\big(\psi_1, \partial_{q_1}\psi_1, \ldots, \partial^{s_1}_{q_1}\psi_1, \psi_2, \partial_{q_2}\psi_2, \ldots, \partial^{s_2}_{q_2}\psi_2, \ldots, \psi_n, \partial_{q_n}\psi_n, \ldots, \partial^{s_n}_{q_n}\psi_n\big). \tag{4.9}$$

This form is nothing but the generalized Wronskian considered by Matveev [1]. Such a uniformity for the KdV equation was also explained by Ma [16] recently.

If we take

$$\psi_j = a_j^+\,\mathrm{e}^{\theta_j} + a_j^-\,\mathrm{e}^{-\theta_j} \tag{4.10}$$

where

$$\theta_j = q_jA + q_j^3B \tag{4.11a}$$

$$A = \frac{1}{2}\mathrm{e}^{-\alpha t}x + \frac{1}{2\alpha}[2\beta(\mathrm{e}^{-3\alpha t}-1) + c_0(\mathrm{e}^{-\alpha t}-1)] \qquad B = \frac{1}{2\alpha}(\mathrm{e}^{-3\alpha t}-1) \tag{4.11b}$$

and a_j^+, a_j^- and q_j are real constants, then it is not difficult to obtain

$$G_m(q_j) = m! \sum_{n=0}^{[\frac{m}{3}]} \sum_{l=0}^{[\frac{m-3n}{3}]} \left\{ \frac{(3q_j)^l B^{l+n}}{(m-2l-3n)!n!l!} \left(A + 3q_j^2 B\right)^{m-2l-3n} \left[a_j^+ \mathrm{e}^{\theta_j} + (-1)^{m+l} a_j^- \mathrm{e}^{-\theta_j}\right] \right\}. \tag{4.12}$$

Now we consider the negatons and positons of equation (1.1). Noting the time dependence of λ described as equation (1.5), we can work out

$$\lambda = -p^2(t) = -(q\,\mathrm{e}^{-\alpha t})^2 \tag{4.13}$$

where q is a constant which will play an important role in the following discussion.

If we take q to be a real number, we can obtain negatons for equation (1.1), where f takes the form of equation (4.9) and

$$\psi_j = \cosh\theta_j \tag{4.14}$$

or

$$\psi_j = \sinh\theta_j. \tag{4.15}$$

The negatons correspond to a negative eigenvalue λ [5] of the Schrödinger spectral problem (1.4*a*).

If we substitute $i\tilde{q}$ for q in equation (4.13) where $\tilde{q}$ is real, i.e. λ is positive, we can obtain positons of equation (1.1) from equation (4.9) where we substitute $\partial_{\tilde{q}_j}^l$ for $\partial_{q_j}^l$ and take

$$\psi_j = \cos\tilde{\theta}_j \tag{4.16}$$

or

$$\psi_j = \sin\tilde{\theta}_j \tag{4.17}$$

in which

$$\tilde{\theta}_j = \tilde{q}_j A - \tilde{q}_j^3 B \tag{4.18}$$

$\tilde{q}_j$ is a real constant, and A and B are defined by equation (4.11*b*)

As given in [5], we have similar classifications for these negatons and positons. The negatons generated from equation (4.9) by choosing equations (4.14) and (4.15) respectively are physically different while the positons generated from equation (4.9) by choosing equations (4.16) and (4.17) respectively are not independent.

Now we give some interesting density graphics of some negatons and positons. We hope this type of graphics can act as a useful tool to describe the behaviours and characteristics of the waves with movable singularities.

Figure 2 corresponds to the one-negaton generated from

$$f = W(\cosh\theta_1, \partial_{q_1}\cosh\theta_1) \tag{4.19}$$

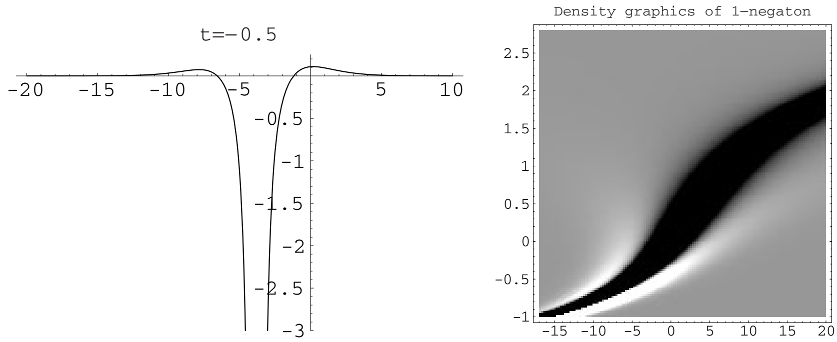
i.e.,

$$u = 8q_1^2\,\mathrm{e}^{-2\alpha t} \frac{\cosh\theta_1(q_1 G_1 \sinh\theta_1 - \cosh\theta_1)}{(2q_1 G_1 + \sinh 2\theta_1)^2} \tag{4.20}$$

where

$$G_1 = A + 3q_1^2 B. \tag{4.21}$$

Figure 2(*b*) is a density image where grey denotes values of u being near zero, the light areas denote the waves with small positive amplitudes, and the dark areas denote the movable singularities where u tends to negative infinity. Comparing with figure 2(*a*), from figure 2(*b*)



(a) (b)

Figure 2. One-negaton as given by equation (4.20) for $q_1 = 0.6, \alpha = 1, \beta = 1$ and $c_0 = 1$. (a) Shape of one-negaton at $t = -0.5$. (b) Density image of one-negaton for $x \in [-17, 20]$ and $t \in [-1, 2.8]$.

we can easily see a continuous singularity track and easily find that the small positive amplitudes decay as t and x increase.

Figure 3 corresponds to the one-negaton generated from

$$f = W(\sinh\theta_1, \partial_{q_1}\sinh\theta_1) \tag{4.22}$$

i.e.,

$$u = -8q_1^2\,\mathrm{e}^{-2\alpha t}\frac{\sinh\theta_1(q_1G_1\cosh\theta_1 - \sinh\theta_1)}{(2q_1G_1 - \sinh 2\theta_1)^2}. \tag{4.23}$$

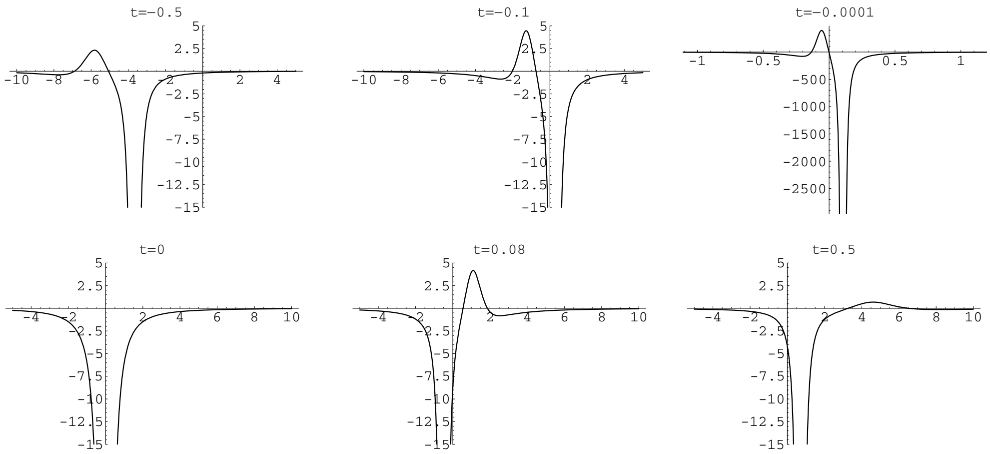
There exist two zeros and one singularity for any non-zero time. We employ a set of graphics in figure 3(a) to show an interesting oscillation of its singularity near $(x, t) = (0, 0)$. This type of oscillation is quite similar to the isospectral case described in [5]. The singularity moves continuously to the right and comes to a momentary halt at a positive value of x. It then reverses its direction of motion, goes past the origin at $t = 0$ with the loss of positive-amplitude wave and again comes to a halt at a negative value of x. Thereafter, this singularity goes continuously in the positive x-direction. Figure 3(b) is the related density image where the dark areas still denote the movable singularities where u tends to $-\infty$, while the light areas, particularly near to $(x, t) = (0, 0)$, denote the waves with large positive amplitudes. Now we give *another* interesting description for the singularity oscillation according to the density graphics. We can suppose that there exist two possible singularities, one is 'real' and the other is 'imaginary'. They are separated by a positive-amplitude wave and seem to be stuck on two sides of the positive-amplitude wave before $t = 0$. They change roles at $(x, t) = (0, 0)$. The 'real' singularity decays rapidly after $t = 0$ and becomes 'imaginary', which corresponds to the dark areas below the light track in figure 3(b). At the same time, the original 'imaginary' singularity changes its role and becomes the 'real' one, which is described as another dark strip above the light areas. We can also find that the positive amplitude decays as t and x increase.

Figure 4 corresponds to the one-positon generated from

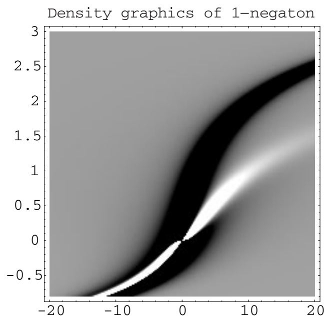
$$f = W(\cos\tilde{\theta}_1, \partial_{\tilde{q}_1}\cos\tilde{\theta}_1) \tag{4.24}$$

i.e.,

$$u = -8\tilde{q}_1^2\,\mathrm{e}^{-2\alpha t}\frac{\cos\tilde{\theta}_1(\tilde{q}_1\tilde{G}_1\sin\tilde{\theta}_1 + \cos\tilde{\theta}_1)}{(2\tilde{q}_1\tilde{G}_1 + \sin 2\tilde{\theta}_1)^2} \tag{4.25}$$

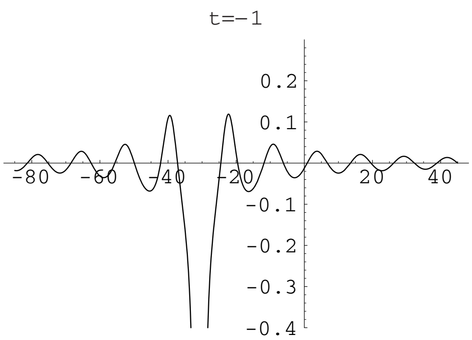


(a)

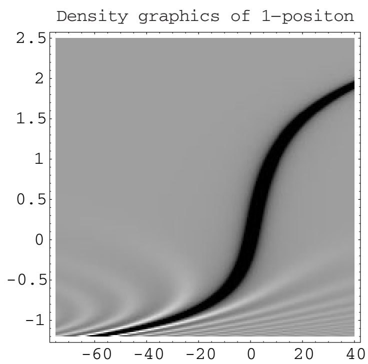


(b)

Figure 3. One-negaton as given by equation (4.23) for $q_1 = 0.6, \alpha = 1, \beta = 1$ and $c_0 = 1$. (a) Shape and motion of one-negaton at different times. (b) Density image of one-negaton for $x \in [-20, 20]$ and $t \in [-0.8, 3]$.

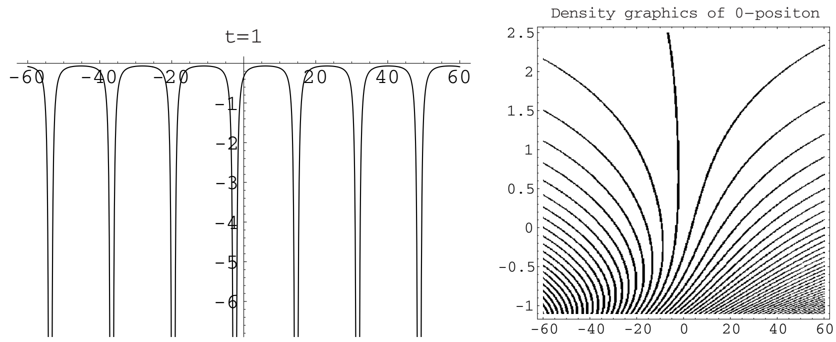


(a)



(b)

Figure 4. One-positon as given by equation (4.25) for $\tilde{q}_1 = 0.1, \alpha = 1.6, \beta = 1$ and $c_0 = 1$. (a) Shape of one-positon at $t = -1$. (b) Density image of one-positon for $x \in [-75, 40], t \in [-1.2, 2.5]$ and plot range of $[-0.3, 0.18]$.



(*a*) (*b*)

Figure 5. Zero-positon as given by equation (4.27) for $\tilde{q}_1 = 1, \alpha = 1, \beta = 1$ and $c_0 = 1$. (*a*) Shape of zero-positon at $t = 1$. (*b*) Density image of zero-positon for $x \in [-60, 60]$ and $t \in [-1.1, 2.5]$.

where

$$\tilde{G}_1 = A - 3\tilde{q}_1^2 B. \tag{4.26}$$

We can clearly see from figure 4(*b*) a beautiful black curve which denotes the continuous singularity track. The black–white stripes at the bottom of figure 4(*b*) describe the wave shapes given as figure 4(*a*). We can also clearly see that these stripes decay as t increases.

We have shown some shapes and behaviours of some negatons and positons by means of density graphics which are very interesting and look like planforms.

Here is an important remark on some so-called positon solutions. Consider the Wronskian (4.8) and substitute $\partial^l_{\tilde{q}_j}$ for $\partial^l_{q_j}$. We let $[\widetilde{\mathcal{C}}^s]$ denote positons generated from equation (4.8) where ψ_j equals $\cos\tilde{\theta}_j$ and we let $[\widetilde{\mathcal{S}}^s]$ denote positons corresponding to $\psi_j = \sin\tilde{\theta}_j$ in equation (4.8). Then, when s is even, the positons possess an infinite number of singularities, while there are only finite singular points for those odd order positons [5]. The infinitely many singularities result in the fact that u does not tend to zero as $x \to \pm\infty$. Remember that when we transform equation (1.1) into the bilinear form (2.1) through equation (2.2), we just employ a condition of $u \to 0$ as $x \to \pm\infty$. That is to say, although $[\widetilde{\mathcal{C}}^{2l}]$ and $[\widetilde{\mathcal{S}}^{2l}]$ solve the bilinear equation (2.1), we have to substitute them directly into equation (1.1) to determine whether they are solutions. For example, the positon $[\widetilde{\mathcal{C}}^0]$

$$u = -\tfrac{1}{2}\tilde{q}_1\,\mathrm{e}^{-2\alpha t}\sec^2\tilde{\theta}_1 \tag{4.27}$$

generated from

$$f = W(\cos\tilde{\theta}_1) = \cos\tilde{\theta}_1 \tag{4.28}$$

can be verified to solve equation (1.1) by a direct substitution. Some characteristics of this solution can be found from figure 5. In figure 5(*b*), the white background denotes values of u near zero and black stripes just denote singularity tracks. For any fixed time t, there are an infinite number of singularities, and the distances between each two neighbouring singularity tracks are the same. This is because in this case $\tilde{\theta}_1$ is a linear function of x. However, the distance between each two neighbouring singularity tracks enlarges exponentially as t increases, which is quite different from the case of the isospectral KdV equation where this distance is invariant with respect to t and the black stripes are parallel to each other.

4.3. Rational-like solutions

To obtain another type of solution in the Wronskian form, we expand the following ψ_j into a power series of q_j

$$\psi_j = \cosh\theta_j = \sum_{m=0}^{+\infty} Q_m(x,t) q_j^{2m} \tag{4.29}$$

where

$$Q_m(x,t) = \sum_{h=0}^{[\frac{2m}{3}]} \frac{A^{2m-3h} B^h}{(2m-3h)!h!}. \tag{4.30}$$

It is not difficult to verify that Q_m satisfies

$$Q_{m,t} = -4Q_{m,xxx} - (c_0 + 6\beta\,\mathrm{e}^{-2\alpha t} + \alpha x) Q_{m,x} \tag{4.31}$$

and

$$Q_{0,xx} = 0 \qquad Q_{m,xx} = \tfrac{1}{4}\,\mathrm{e}^{-2\alpha t} Q_{m-1} \qquad (m \geqslant 1). \tag{4.32}$$

Then, according to theorem 1, we know immediately that the Wronskian

$$f = W(Q_0, Q_1, \ldots, Q_n) \tag{4.33}$$

solves the bilinear equation (2.1).

The solution generated from equation (4.33) is a rational-like solution. In fact, employing the transformation

$$T = \mathrm{e}^{-\alpha t} \tag{4.34}$$

the KdV equation with loss and non-uniformity terms (1.1) can be written as

$$-\alpha T u_T + 2\alpha u + (c_0 + 6\beta T^2 + \alpha x) u_x + 6 u u_x + u_{xxx} = 0. \tag{4.35}$$

Then, equation (4.33) can denote a rational solution of the above equation.

We note that following the procedures given in [9] and [10], we can verify that equation (4.33) and

$$f = (\psi_1, \psi_2, \ldots, \psi_l, Q_0, Q_1, \ldots, Q_{n-l})$$

satisfy equation (2.1) by taking the limitations of $(q_1, \ldots, q_{n+1}) \to (0, \ldots, 0)$ and $(q_{l+1}, \ldots, q_{n+1}) \to (0, \ldots, 0)$, respectively.

Another rational-like solution corresponding to

$$\psi_j = \sinh\theta_j \tag{4.36}$$

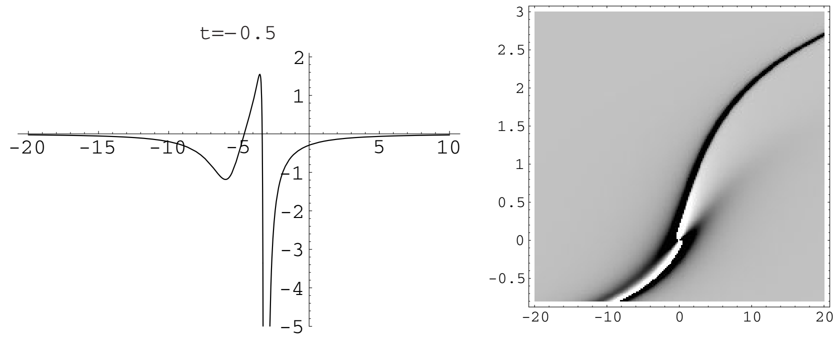
can be given by

$$f = W(\bar{Q}_0, \bar{Q}_1, \ldots, \bar{Q}_n) \tag{4.37}$$

where

$$\bar{Q}_m(x,t) = \sum_{h=0}^{[\frac{2m+1}{3}]} \frac{A^{2m+1-3h} B^h}{(2m+1-3h)!h!}. \tag{4.38}$$

We note that, if $\left|a_j^+\right| \neq |a_j^-|$ in (4.10), then equation (4.10) only generates a trivial rational-like solution for equation (1.1).



(*a*) (*b*)

Figure 6. Rational-like solution as given by equation (4.40) for $\tilde{q}_1 = 0.6, \alpha = 1, \beta = 1$ and $c_0 = 1$. (*a*) Shape of rational-like solution at $t = -0.5$. (*b*) Density image of rational-like solution for $x \in [-20, 20]$, $t \in [-0.8, 3]$ and plot range of $[-1.6, 0.5]$.

Here are the first two solutions generated from equation (4.37):

$$u = -\frac{\mathrm{e}^{-2\alpha t}}{2A^2} \tag{4.39}$$

and

$$u = -\frac{3}{2} A\, \mathrm{e}^{-2\alpha t} \frac{A^3 - 2B}{(A^3 - 3B)^2}. \tag{4.40}$$

An interesting fact is that the rational-like solution (4.40) shows quite similar behaviours to one-negaton as given by equation (4.23). Both possess two zeros (for any non-zero time), one singularity and singularity oscillations near $(x, t) = (0, 0)$. The graphics of equation (4.40) are given in figure 6 where we employ a special plot range so that we can more clearly find the changes of two separated singularity strips.

We can also consider the rational-like solutions generated by $\cos\tilde{\theta}_j$. However, it is easy to find that

$$\cos\tilde{\theta}_j = \sum_{m=0}^{+\infty} (-1)^m Q_m(x, t) \tilde{q}_j^{2m}. \tag{4.41}$$

That is to say, although as seeds $\cosh\theta_j$ generates negatons while $\cos\tilde{\theta}_j$ generates positons, they lead to the same rational-like solutions. There are similar results for $\sinh\theta_j$ and $\sin\tilde{\theta}_j$ due to

$$\sin\tilde{\theta}_j = \sum_{m=0}^{+\infty} (-1)^m \bar{Q}_m(x, t) \tilde{q}_j^{2m+1}. \tag{4.42}$$

4.4. Mixed solutions and interactions

As we have obtained many functions which satisfy the conditions (3.5) and (3.6), we can obtain abundant solutions of equation (1.1). Let us first define

$$\begin{aligned} &\varphi_j = \mathrm{e}^{\theta_j} + (-1)^{j-1}\mathrm{e}^{\theta_j} \qquad &&\psi_j = \cosh\theta_j \qquad &&\bar{\psi}_j = \sinh\theta_j \\ &\chi_j = \cos\tilde{\theta}_j &&\bar{\chi}_j = \sin\tilde{\theta}_j && \end{aligned} \tag{4.43}$$

where θ_j and $\tilde{\theta}_j$ are given by equations (4.11) and (4.18) respectively, and $j = 1, 2, \ldots$. Then we give a quite general Wronskian which solves the bilinear equation (2.1)

$$\begin{aligned} f = W(&\varphi_1, \varphi_2, \ldots, \varphi_n \psi_1, \partial_{q_1}\psi_1, \ldots, \partial_{q_1}^{s_1}\psi_1, \ldots, \psi_h, \partial_{q_h}\psi_h, \ldots, \partial_{q_h}^{s_h}\psi_h, \\ &\bar{\psi}_1, \partial_{q_1}\bar{\psi}_1, \ldots, \partial_{q_1}^{k_1}\bar{\psi}_1, \ldots, \bar{\psi}_i, \partial_{q_i}\bar{\psi}_i, \ldots, \partial_{q_i}^{k_i}\bar{\psi}_i, \chi_1, \partial_{\tilde{q}_1}\chi_1, \ldots, \partial_{\tilde{q}_1}^{l_1}\chi_1, \\ &\ldots, \chi_r, \partial_{\tilde{q}_r}\chi_r, \ldots, \partial_{\tilde{q}_r}^{l_r}\chi_r, \bar{\chi}_1, \partial_{\tilde{q}_1}\bar{\chi}_1, \ldots, \partial_{\tilde{q}_1}^{p_1}\bar{\chi}_1, \ldots, \bar{\chi}_s, \partial_{\tilde{q}_s}\bar{\chi}_s, \ldots, \partial_{\tilde{q}_s}^{p_s}\bar{\chi}_s, \\ &Q_0, \ldots, Q_{m_1}, \bar{Q}_0, \ldots, \bar{Q}_{m_2}) \end{aligned} \tag{4.44}$$

where Q_j and $\bar{Q}_j$ are given by equations (4.30) and (4.38), respectively.

We can also consider some special reductions to obtain various solutions in the Wronskian form through omitting properly some functions and their related derivatives from the above Wronskian. These solutions show various interactions of solitons, negatons, positons and rational-like solutions. For example, the Wronskian

$$\begin{aligned} f = W\big(&\psi_1, \partial_{q_1}\psi_1, \ldots, \partial_{q_1}^{s_1}\psi_1, \ldots, \psi_h, \ldots, \\ &\partial_{q_h}^{s_h}\psi_h, \bar{\psi}_1, \partial_{q_1}\bar{\psi}_1, \ldots, \partial_{q_1}^{k_1}\bar{\psi}_1, \ldots, \bar{\psi}_n, \ldots, \partial_{q_n}^{k_n}\bar{\psi}_n\big) \end{aligned} \tag{4.45}$$

describes negaton–negaton scattering, and

$$f = W\big(\psi_1, \partial_{q_1}\psi_1, \ldots, \partial_{q_1}^{s_1}\psi_1, \ldots, \psi_h, \ldots, \partial_{q_h}^{s_h}\psi_h, Q_0, \ldots, Q_m\big) \tag{4.46}$$

can show the interaction of negatons and the rational-like solution.

We have obtained some generalized Wronskian solutions, such as negatons, positons, rational-like solutions and the mixed solutions which can show various interactions of the solutions. It is obvious that the arbitrary constant q_j plays a key role in the processes of finding these solutions.

5. The infinitely many conservation laws

Although the KdV equation with loss and non-uniformity terms (1.1) is an equation with non-isospectral properties, we can derive its infinitely many conservation laws.

According to equation (1.5), we take

$$\lambda = -q^2\,\mathrm{e}^{-2\alpha t} \qquad (q \in R). \tag{5.1}$$

Then the Lax pair (1.4) can be rewritten as

$$\phi_{xx} = -u\phi - q^2\,\mathrm{e}^{-2\alpha t}\phi \tag{5.2a}$$

$$\phi_t = -u_x\phi - (-4q^2\,\mathrm{e}^{-2\alpha t} + 2u + c_0 + 6\beta\,\mathrm{e}^{-2\alpha t} + \alpha x)\phi_x. \tag{5.2b}$$

As far as the new constant q is concerned, equation (1.1) is an isospectral equation.

Taking

$$\omega(x, t, q) = \frac{\phi_x}{\phi} + \mathrm{i}\,q\,\mathrm{e}^{-\alpha t} \tag{5.3}$$

we obtain the Riccati equation related to equation (5.2a), i.e.,

$$2\mathrm{i}q\omega = \mathrm{e}^{\alpha t}(\omega_x + \omega^2 + u) \tag{5.4}$$

which can be solved by

$$\omega(x, t, q) = \sum_{j=1}^{+\infty} \frac{\omega_j(x, t, q)}{(2\mathrm{i}q)^j} \tag{5.5a}$$

with

$$\omega_{j+1} = u\,\mathrm{e}^{\alpha t}\delta_{j,0} + \mathrm{e}^{\alpha t}\left(\omega_{j,x} + \sum_{i=1}^{j-1}\omega_i\omega_{j-i}\right). \tag{5.5b}$$

Then, similar to the isospectral case, the conservation laws can be given by

$$(\omega - \mathrm{i}q\,\mathrm{e}^{-\alpha t})_t = \left[-u_x - (-4q^2\,\mathrm{e}^{-2\alpha t} + 2u + c_0 + 6\beta\,\mathrm{e}^{-2\alpha t} + \alpha x)(\omega - \mathrm{i}q\,\mathrm{e}^{-\alpha t})\right]_x$$

i.e.,

$$\omega_t = \left\{(2\mathrm{i}q)\,\mathrm{e}^{-\alpha t}u - u_x - \left[(2\mathrm{i}q)^2\,\mathrm{e}^{-2\alpha t} + 2u + c_0 + 6\beta\,\mathrm{e}^{-2\alpha t} + \alpha x\right]\omega\right\}_x \tag{5.6}$$

where we have made use of $(\ln\phi)_{xt} = (\ln\phi)_{tx}$. Noticing that

$$\omega_R = -\tfrac{1}{2}[\ln(\omega_I - q\mathrm{e}^{-\alpha t})]_x$$

where ω_R and ω_I denote the real and imaginary parts of ω respectively, i.e., $\omega = \omega_R + \mathrm{i}\omega_I$, the non-trivial infinitely many conservation laws for the KdV equation with loss and non-uniformity terms are given by

$$\omega_{2j+1,t} = -\left[\mathrm{e}^{-2\alpha t}\omega_{2j+3} + (2u + c_0 + 6\beta\,\mathrm{e}^{-2\alpha t} + \alpha x)\omega_{2j+1}\right]_x \qquad (j = 0, 1, \ldots) \tag{5.7}$$

where each ω_{2j+1} is a non-trivial conserved density.

6. Conclusion

The KdV equation with loss and non-uniformity terms has been solved using the Hirota method and Wronskian technique. We have shown that the procedure proposed by Sirianunpiboon *et al* [15] can also apply to this non-isospectral equation. Solitons, negatons, positons, rational-like solutions and mixed solutions of this equation are obtained, and the mixed solutions can show various interactions. The negatons and positons correspond, respectively, to negative and positive eigenvalues of the Schödinger spectral problem (1.4*a*). The characteristics of these solutions with non-isospectral properties have been described through some graphics made by using Mathematica. Two graphics have been given to describe the shape and motion of the so-called one-soliton. It is shown that the amplitude and velocity of the solitary wave vary with time t. This may be a typical characteristic of 'solitons' with non-isospectral properties. In general, for an isospectral equation, for instance, the KdV equation, the amplitude and velocity of a soliton are determined by the spectral (eigenvalue) of the Schrödinger spectral problem. For equation (1.1), we have shown that the amplitude of a solitary wave is also quite closely related to the eigenvalue λ given by equation (1.5). Equation (1.1) can describe soliton waves in a certain type of non-uniform media, and α can control the decreasing rate of amplitudes. Besides, both β and c_0 can affect motion velocities. We have also given some interesting density graphics of negatons, positons and rational-like solutions. Taking advantage of the density graphics we can easily see the singularity tracks and the decays of some amplitudes, and we have found differences between these solutions with singularities. In addition, some differences between isospectral and non-isospectral solutions have been described. We have also given a new and interesting explanation for the singularity oscillations of one-negaton (4.23) and rational-like solution (4.40) according to their density graphics. To sum up, we have obtained a variety of solutions and an infinite number of conservation laws for the KdV equation with loss and non-uniformity terms, an evolution equation with non-isospectral properties. We hope that the discussion in this paper will be useful for the study of other non-isospectral systems.

Acknowledgments

This project is supported by the National Natural Science Foundation of China and the Special Funds for Major Specialities of Shanghai Education Committee.

References

[1] Matveev V B 1992 *Phys. Lett.* A **166** 205–8, 209–12
Matveev V B 1992 *Phys. Lett.* **168** 463
[2] Matveev V B 1994 *J. Math. Phys.* **35** 2955–70
[3] Beutler R 1993 *J. Math. Phys.* **34** 3098–109
[4] Stahlhofen A A and Matveev V B 1995 *J. Phys. A: Math. Gen.* **28** 1957–65
[5] Rasinariu C, Sukhatme U and Khare A 1996 *J. Phys. A: Math. Gen.* **29** 1803–23
[6] Zeng Y B, Shao Y J and Xue W M 2003 *J. Phys. A: Math. Gen.* **36** 5035–43
[7] Hirota R 1971 *Phys. Rev. Lett.* **27** 1192–94
[8] Freeman N C and Nimmo J J C 1983 *Phys. Lett.* A **95** 1–3
[9] Nimmo J J C and Freeman N C 1983 *Phys. Lett.* A **96** 443–6
[10] Wu H and Zhang D J 2003 *J. Phys. A: Math. Gen.* **36** 4867–73
[11] Ablowitz M J and Satsuma J 1978 *J. Math. Phys.* **19** 2180–86
[12] Zhang D J 2002 *J. Phys. Soc. Japan* **71** 2649–56
[13] Zhang D J and Chen D Y 2003 *Physica* A **321** 467–81
[14] Zhang D J 2003 *Chaos Solitons Fractals* **18** 31–43
[15] Sirianunpiboon S, Howard S D and Roy S K 1988 *Phys. Lett.* A **134** 31–3
[16] Ma W X 2003 *Chaos Solitons Fractals* **19** 163–70
[17] Hirota R and Satsuma J 1976 *J. Phys. Soc. Japan* **41** 2141–42
[18] Chan W L and Li K S 1989 *J. Math. Phys.* **30** 2521–26
[19] Chen H H and Liu C S 1976 *Phys. Rev. Lett.* **37** 693–7
[20] Bogoyavlensky O I 1991 *Oprokidiwayushiesya Solitoni (Overturning Solitons)* (Moscow: Nauka)
[21] Newell A C 1979 *Proc. R. Soc.* A **365** 283–311